Linear Algebra II

06/05/2014, Monday, 18:30-21:30

You are **NOT** allowed to use any type of calculators.

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1 (15 pts) **Gram-Schmidt process**

Consider the vector space $\mathbb{R}^3$ with the inner product

$$\langle x, y\rangle = x^T y.$$

Apply the Gram-Schmidt process to transform the basis

$$\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\}$$

into an orthonormal basis.

2 (15 pts) **Cayley-Hamilton theorem**

Consider the matrix

$$M = \begin{bmatrix}1 & 1 & -1\\-1 & 1 & 1\\0 & 2 & 0\end{bmatrix}.$$

By using Cayley-Hamilton theorem, show that $M^k = 2^{k-2}M^2$ for any $k \geqslant 2$.

3 (12+3=15 pts) **Singular value decomposition**

Consider the matrix

$$M = \begin{bmatrix}2 & 1\\1 & 2\\2\sqrt{2} & 2\sqrt{2}\end{bmatrix}.$$

(a) Find a singular value decomposition for M.

(b) Find the best rank 1 approximation of M.

4 (3+3+3+3+3=15 pts) **Eigenvalues and singular values**

Let $J \in \mathbb{R}^{n\times n}$ be a skew-symmetric matrix, that is $J + J^T = 0$.

(a) Show that the real part of any eigenvalue of J must be zero.

(b) Show that J is singular if n is an odd number.

(c) Show that J is unitarily diagonalizable.

(d) Show that if λ is an eigenvalue of J then $\sqrt{-\lambda^2}$ is a singular value of J.

(e) Show that if σ is a singular value of J then $i\sigma$ is an eigenvalue of J.

5 (7+8=15 pts) **Positive definiteness**

(a) Consider the function

$$f(x,y,z) = -\frac{1}{4}\left(\frac{1}{x^4} + \frac{1}{y^4} + \frac{1}{y^4}\right) + yz - x - 2y - 2z.$$

Show that $(1,1,1)$ is a stationary point and determine whether this stationary point is local minimum/maximum or saddle point.

(b) Let

$$M = \begin{bmatrix} a+1 & a & 0 \\ a & a+1 & a \\ 0 & a & a \end{bmatrix}$$

where a is a real number. Determine all values of a for which M is

(i) positive definite.

(ii) negative definite.

6 (2+3+10=15 pts) **Jordan canonical form**

Consider the matrix

$$\begin{bmatrix} 0 & 0 & -1 \\ -2 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix}.$$

(a) Find its eigenvalues.

(b) Is it diagonalizable? Why?

(c) Put it into the Jordan canonical form.

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10 pts free